Aggregation Operator Based Fuzzy Pattern Classifier Design

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***Abstract*—This paper presents a novel modular fuzzy pattern classifier design framework for intelligent automation systems, developed on the base of the established Modified Fuzzy Pattern Classifier (MFPC) and that allows designing novel classifier models which are hardware-efficiently implementable. The performances of novel classifiers using substitutes of MFPC's geometric mean aggregator are benchmarked in the scope of an image processing application against the MFPC to reveal classification improvement potentials for obtaining higher classification rates.**

***Index Terms*—Fuzzy systems, image processing, fuzzy pattern classification, automation, machine learning, artificial intelligence, pattern recognition.**

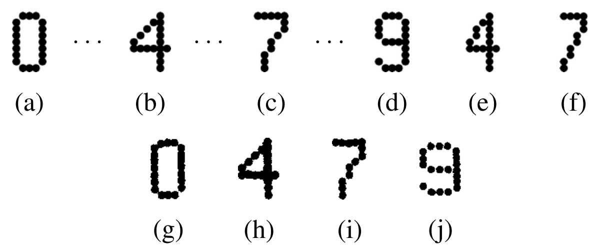

Fig. 1. Sample training images of twelve digit classes (0 – 9 (cf. *(a) – (d)*) plus modified 4 and 7 (cf. *(e) and (f)*)). Figures *(g) – (j)* show samples of the test images.

I. Introduction

Automation techniques such as image processing or machine vision—or generally speaking *signal processing*—and pattern recognition are applied in industrial production processes (e.g. for quality inspection) as well as in consumer products like compact digital cameras (e.g. for face recognition) and are getting more and more important these days. The reasons are diverse, but mainly processes are automated to reduce costs, improve production quality, handle large amounts of data or just for the sake of convenience.

In order to bring *"artificial intelligence"* into automation systems, they need to gain knowledge about their applications by human-centric learning techniques, often incorporating *Fuzzy Logic* (based on Lotfi A. Zadeh's work about *Fuzzy Set Theory* [1]) to represent the information and make decisions based on them.

Established as decision-making instrument in the field of industrial automation processes is the *Modified Fuzzy Pattern Classifier (MFPC)* introduced by Lohweg et.al. [2]. It has proven its performance and robustness in this area, even in applications where the inputs are noisy or vary in their values. The focus of this paper lies on industrial image processing applications where certainty in decisions and real-time demands are critical. The latter implies hardware-efficient implementations which is fulfilled by the MFPC. Although, the MFPC model is well suited for a lot of applications, it is questioned if this models is actually the *best* for these applications.

This paper introduces a novel modular classifier design framework based on the principles of the MFPC. It can be used to develop and apply classifiers—which retain to be easy and efficiently implementable—to achieve the best performance for a specific application. A series of novel classifier models are designed and their performances are benchmarked against the performance of the original MFPC in the scope of an image processing application to disclose MFPC's improvement potentials.

The application is about optical character recognition where the printing of dot-matrix printed digits is checked during a production process. The test case includes images taken from twelve classes of digits, being all digits from "0" to "9" plus one class of modified "4" and "7", respectively. A total of 746 images (or objects) to be classified are used during the test phase. Which class an image belongs to is unknown to the classifier, but known to the user. Compared to the images used to train the classifier (cf. Fig. 1 (a) – (f)), the test images (cf. Fig. 1 (g) – (j)) are not so homogeneous due to distortions during printing, which gives an impression how well a classifier needs to perform and how robust against disturbances it must be.

The paper's structure is as follows: Section II introduces the concepts of Fuzzy Pattern Classification the MFPC relies on. In Sect. III the aggregation operators used to substitute MFPC's aggregation operator are described before the hereby created novel classifiers are evaluated and benchmarked against the MFPC in Sect. IV. The paper ends with Sect. V by providing a conclusion and an outlook.

II. Fuzzy Pattern Classification

Fuzzy Pattern Classifiers (FPC) [3], which were originally developed by Bocklisch, basically "look" at an object and sort this object into a class known to the classifier. This is done by comparing a feature vector $\mathbf{m}$, extracted from an object under inspection, to features of a typical member of a class. In fact, while $\mathbf{m} = (m_1, m_2, \ldots, m_M)$, $m_i \in I \ \forall i$, is a vector of M crisp values, the features of a typical member of one class are fuzzy variables represented by membership functions $\mu_i : I \to I \ \forall i \in \mathbb{N}_M$, where I denotes the closed unit interval of real numbers $[0, 1]$ and $\mathbb{N}_M$ the set of natural numbers $\{1, 2, \ldots, M\}$. Each extracted feature m_i of the feature vector $\mathbf{m}$ serves as input for the membership functions. The resulting membership values $\mu_i(m_i)$ can be interpreted as similarity of feature m_i compared to the same feature of the typical class member. The membership values are then aggregated by the classifier to result in one single membership value $\mu(\mathbf{m}) \in I$ for the complete object under inspection. In formal descriptions of a classifier, both parts—the fuzzy membership functions and the aggregation part—may not appear explicitly. The formal description of the Modified Fuzzy Pattern Classifier, for example, looks like a single membership function (cf. (3)), nevertheless also the aggregation part is present.

A. Modified Fuzzy Pattern Classifier

Lohweg's *Modified Fuzzy Pattern Classifier (MFPC)* [2] is—as its name implies—a derivate of Bocklisch's Fuzzy Pattern Classifier (FPC), optimized for hardware implementations. MFPC's general concept of simultaneously aggregating an arbitrary number of unimodal membership functions to compute an overall membership value is borrowed from the original FPC. Lohweg's intention—leading to the MFPC in the form of (3)—was to create a pattern recognition system on a Field Programmable Gate Array (FPGA) that can be applied in high-speed industrial applications [2]. This section introduces the MFPC and derives its properties, which are the base for all further investigations.

The fuzzy membership function $\mu : I \to I$ used for the MFPC is Eichhorn's parameterizable unimodal potential function [4]

$$\mu_{\mathrm{MFPC}}(m, \mathbf{p}) = 2^{-d(m,\mathbf{p})} \in I \quad (1)$$
$$\text{with } d(m, \mathbf{p}) = \left(\frac{|m - S|}{C}\right)^{D}.$$

$\mathbf{p} = (S, C, D)$ is a parameter vector defining the membership function's properties, namely mean value (S), width (C) and steepness of its edges (D), which increases with increasing D. $d(m, \mathbf{p})$ computes the distance or dissimilarity of the feature m to the membership function's mean value S, thus to an ideal member of a class. A sample MFPC membership function is depicted in Fig. 2.

The MFPC membership function's parameters are obtained automatically during a learning phase. During this phase, the features m_i are extracted from N typical members of a class. For each feature, the parameters are then calculated as [2]

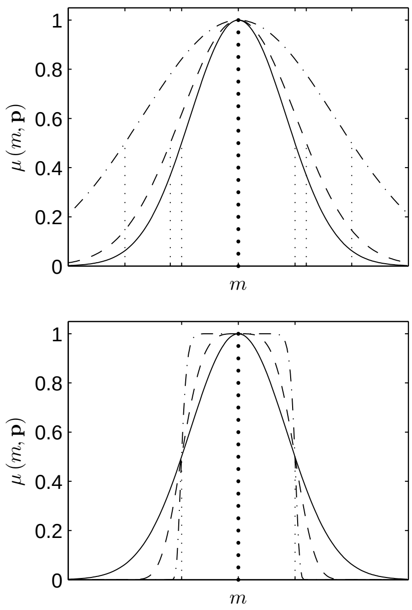

Fig. 2. Sample MFPC membership function at $D = 2$ and $p_{C_e} = 0$ (*solid*). The upper and lower plot show changes (*dashed* → *dash-dotted*) with increasing p_{C_e} and D, respectively. The vertical *dotted* line shows respective $S \pm C$, the *bold-dotted* line S.

$$S = \Delta + m_{min}, \quad C = (1 + 2p_{c_e}) \cdot \Delta \quad (2)$$

with the arbitrary width adjustment factor $p_{c_e} \in I$ (called *percental elementary fuzziness*) and where

$$m_{max} = \max_{i=1}^{N} m_i, \ m_{min} = \min_{i=1}^{N} m_i, \ \Delta = \frac{m_{max} - m_{min}}{2}.$$

D is an arbitrarily chosen integer for which Lohweg proposes a power of 2 to keep calculating the distance measure $d(m, \mathbf{p})$ hardware-efficient, typically limited to $D \in \{2, 4, 8, 16\}$.

MFPC's ability to simultaneously calculate membership values and aggregate them is expressed as an aggregation function $h : I^n \to I$ by [2]

$$h_{\mathrm{MFPC}}(\mathbf{m}, \mathbf{p}) = 2^{-\frac{1}{M}\sum_{i=1}^{M} d_i(m_i, \mathbf{p}_i)} \quad (3)$$
$$\text{with } d_i(m_i, \mathbf{p}_i) = \left(\frac{|m_i - S_i|}{C_i}\right)^{D_i}$$

for M different features. The feature m in (1) has actually become a feature vector $\mathbf{m}$, and $\mathbf{p}$ is no more only a vector but a matrix of parameter vectors $\mathbf{p}_i$, parameterizing each membership function belonging to every feature m_i. Since the MFPC aggregates fuzzy membership functions, it can be seen as a *fuzzy aggregation operator*. Its properties are derived in the following.

1) MFPC as Averaging Operator.: To show the aggregation character of the MFPC, its formal description (3) is rewritten to

$$h_{\text{MFPC}}(\mathbf{m},\mathbf{p}) = \left(\prod_{i=1}^{M} 2^{-d_i(m_i,\mathbf{p}_i)}\right)^{\frac{1}{M}} = \left(\prod_{i=1}^{M} \mu_{\text{MFPC},i}(m_i,\mathbf{p}_i)\right)^{\frac{1}{M}}. \quad (4)$$

Equation (4) shows that MFPC aggregates its membership functions $\mu_{\text{MFPC},i}$ using the *geometric mean* operator defined as $h_{\text{GM}}(a_1, a_2, \ldots, a_n) = (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}$. The geometric mean actually is an averaging operator, thus also the MFPC is an averaging operator. Equation (4) shows additionally that MFPC's aggregation operator can be isolated and therefore be substituted by any other operator. Nevertheless, to retain the classifier's aggregation character of being an *averaging* operator, the aggregation operator's substitute must be an averaging operator, too. The investigated substitutes are described in Sect. III.

2) MFPC's Andness and Orness.: Averaging operators are situated between the *min* (or *pure AND*) operator and the *max* (or *pure OR*) operator [5]. Depending on the chosen averaging operator, its aggregation behavior may be more *AND*like or more *OR*like. This characteristic behavior can be expressed by the degrees of *andness* ρ and *orness* ω, respectively.

Dujmović' measures of *global* andness ρ_g and orness ω_g for a general averaging operator $h(\mathbf{a})$ with $\mathbf{a} \neq (0, \ldots, 0)$ are defined as [6]

$$\rho_g^{h(\mathbf{a})} = \frac{\overline{\max(\mathbf{a})} - \overline{h(\mathbf{a})}}{\overline{\max(\mathbf{a})} - \overline{\min(\mathbf{a})}}, \quad \omega_g^{h(\mathbf{a})} = 1 - \rho_g^{h(\mathbf{a})} \quad (5)$$

where $\overline{g(\mathbf{a})}$ is the expected value of $g(\mathbf{a})$, defined by $\overline{g(\mathbf{a})} = \int_{I^n} g(\mathbf{a}) d\mathbf{a}$ for $\mathbf{a} \in I^n$.

Dujmović and Larsen applied (5) and found geometric mean's global andness for $n = 2$ and $n \to \infty$—thus its boundaries—to be $\rho_g^{h_{\text{GM}}}(2) = \frac{2}{3} \approx 0.667$ and $\lim_{n\to\infty} \rho_g^{h_{\text{GM}}}(n) = 1 - \frac{1}{e} \approx 0.632$, respectively [6].

III. Fuzzy Aggregation Operators

The preceding Sect. II showed that the MFPC aggregates using the geometric mean aggregation operator. Due to its fixed andness of around $\frac{2}{3}$ for a fixed number of parameters, the MFPC is less flexible than e.g. Yager's class of *Ordered Weighted Averaging (OWA)* operators [7]. Since the aggregation operator is isolated in MFPC's re-definition (4), it is possible to substitute it with a different aggregation operator. Two substitute candidates (amongst many others) are the aforementioned class of OWA operators and also Larsen's class of *Andness-directed Importance Weighting Averaging (AIWA)* operators [8]. Both are parameterizable to have a desired andness and both do not have geometric mean's mandatory property [6] resulting in an aggregated value of zero or close to zero if at least one of the values to be aggregated is zero or close to it, which might be problematic in classification applications. The operators are briefly described in the following subsections.

A. Ordered Weighted Averaging Operator

Yager introduces a class of aggregation operators called Ordered Weighted Averaging (OWA) operators [7]. The aggregation of n features $\mathbf{a} = (a_1, \ldots, a_n)$ is defined by

$$h_{\text{OWA}}(\mathbf{w},\mathbf{a}) = \sum_{i=1}^{n} \left(w_i \cdot a_{(i)}\right), \quad (6)$$

where $\mathbf{w} = (w_1, \ldots, w_n)$ is a vector of positional weights, called *OWA weights*, with $w_i \in I$ and $\sum_{i=1}^{n} w_i = 1$; $(\cdot)$ denotes a permutation of $\mathbf{a}$ on $\mathbb{N}_n$ such that $a_{(1)} \geq \cdots \geq a_{(n)}$, thus sorts the vector's elements in descending order.

$\mathbf{w}$ determines the operator itself, thus the operator's andness $\rho_g^{h_{\text{OWA}}}$. The OWA weights w_i can be obtained by applying a regular increasing continuous *quantifier function* $\varphi : I \to I$ satisfying *(1)* $\varphi(0) = 0$, *(2)* $\varphi(1) = 1$ and *(3)* $x_1 < x_2 \Rightarrow \varphi(x_1) \leq \varphi(x_2)$. For dimension $n \geq 2$, the OWA weights are obtained by $w_i = \varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)$, $i \in \mathbb{N}_n$ [9]. One class of functions satisfying the quantifier functions' conditions is the class of *regular monotonic quantifiers*, defined by $\varphi_\beta(x) = x^\beta$ [9]. Thus, the OWA weights are obtained by

$$w_i = \left(\frac{i}{n}\right)^\beta - \left(\frac{i-1}{n}\right)^\beta, \quad i \in \mathbb{N}_n \quad (7)$$

$$\text{with } \beta = \frac{\rho_Q}{1-\rho_Q} \in [0,\infty), \quad \rho_Q \in [0,1],$$

$$\text{with } \rho_g^{h_{\text{OWA}}}(\mathbf{w}) = 1 - \left(\frac{1}{n-1}\sum_{i=1}^{n} \left[(n-i) \cdot w_i\right]\right), \quad (8)$$

where ρ_Q, called *quantifier andness*, is an estimator of the OWA operator's (real) andness $\rho_g^{h_{\text{OWA}}}(\mathbf{w})$ [7]; notice that (8) is completely consistent to (5) [6].

There is actually a difference between ρ_Q and $\rho_g^{h_{\text{OWA}}}$ due to ρ_Q's independence from dimension n, while $\rho_g^{h_{\text{OWA}}}$—defined by $\mathbf{w}$—depends on n (cf. Equation (8)). This demands adjustments of ρ_Q to obtain an OWA operator with the desired (real) andness $\rho_g^{h_{\text{OWA}}}$.

B. Andness-Directed Importance Weighting Averaging Operator

The class of Andness-directed Importance Weighted Averaging (AIWA) operators by Larsen [8] is based on the power means [10] and extends these to incorporate also importance weighting of the arguments. AIWA aggregation of $\mathbf{a} = (a_1, \ldots, a_n)$ is defined as

$$h_{\text{AIWA}}(\mathbf{v},\mathbf{a}) = \begin{cases} \left(\frac{\sum_{i=1}^{n}(v_i \cdot a_i)^\gamma}{\sum_{i=1}^{n}(v_i)^\gamma}\right)^{\frac{1}{\gamma}} & \rho \in \left(0, \frac{1}{2}\right] \\ 1 - \left(\frac{\sum_{i=1}^{n}(v_i \cdot (1-a_i))^{\frac{1}{\gamma}}}{\sum_{i=1}^{n}(v_i)^{\frac{1}{\gamma}}}\right)^\gamma & \rho \in \left[\frac{1}{2}, 1\right) \end{cases}, \quad (9)$$

$$\text{where } \gamma = \frac{1}{\rho} - 1,$$

with $\mathbf{v} = (v_1, \ldots, v_n)$ being a vector containing importance weights which satisfy $v_i \in I$ and is maximum-normalized to 1, i.e. $\max_{i=1}^{n} v_i = 1$. If weighting is not required, an

TABLE I

CLASSIFICATION RATES r_+ FOR EACH CLASSIFIER AT ANDNESS DEGREES ρ_g BY THE OPERATOR'S PARAMETERS ρ_Q/ρ WITH REGARD TO MFPC PARAMETERS D AND p_{C_e}.

ρ_g	Classifier	ρ_Q/ρ	$D=2$ p_{C_e}	$D=2$ r_+	$D=4$ p_{C_e}	$D=4$ r_+	$D=8$ p_{C_e}	$D=8$ r_+	$D=16$ p_{C_e}	$D=16$ r_+
0.500	$_{\text{OWA}}h_{\text{MFPC}}$	0.500	0.370	84.58%	0.370	87.80%	0.310	92.36%	**0.290**	**92.90%**
	$_{\text{AIWA}}h_{\text{MFPC}}$	0.500	0.370	84.58%	0.370	87.80%	0.310	92.36%	**0.290**	**92.90%**
0.560	$_{\text{OWA}}h_{\text{MFPC}}$	0.557	0.405	84.58%	0.370	87.94%	0.305	92.49%	0.275	92.76%
	$_{\text{AIWA}}h_{\text{MFPC}}$	0.630	0.240	85.92%	0.270	90.21%	**0.245**	**93.16%**	0.265	92.76%
0.637	h_{MFPC}	—	0.155	81.77%	0.445	82.17%	0.755	82.44%	1.000	82.44%
	$_{\text{OWA}}h_{\text{MFPC}}$	0.630	0.215	84.45%	0.355	88.74%	0.305	92.63%	**0.275**	**92.76%**
	$_{\text{AIWA}}h_{\text{MFPC}}$	0.750	0.135	85.52%	0.185	89.95%	0.270	89.95%	0.315	89.95%
0.700	$_{\text{OWA}}h_{\text{MFPC}}$	0.689	0.280	84.45%	0.335	88.87%	**0.295**	**92.63%**	**0.275**	**92.63%**
	$_{\text{AIWA}}h_{\text{MFPC}}$	0.820	0.430	82.71%	0.795	82.57%	1.000	82.31%	1.000	79.09%

unweighted AIWA operator is achieved by $v_i = 1\ \forall i$. To estimate an operator with a desired andness degree ρ, the operator's parameter γ is calculated by its equation given in (9). As with OWA operators, AIWA operator's real andness ρ_g^{AIWA}, as defined by (5), is dependent on dimension n and thus different from the desired, dimensionless andness ρ which implies adjustments of γ until the desired andness ρ_g^{AIWA} is achieved.

IV. CLASSIFIER EVALUATION

The novel classifier models incorporate OWA and AIWA operators, respectively, and are generated in a straightforward way. Each argument a_i is substituted by an MFPC membership function $\mu_{\text{MFPC},i}$. Both classifier models, as applied here, are nothing else than fuzzy mean operators since no importance weighting is used. They are benchmarked against the performance of the original MFPC applied in the scope of the optical character recognition application described in Sect. I. For this benchmark, all classifier models are implemented, executed and evaluated in MATLAB using a classification framework where the aggregation operator can be substituted easily.

Formally, the classifiers are expressed for the OWA operators and for the unweighted AIWA operators ($\Rightarrow v_i = 1\ \forall i$), respectively, by

$$_{\text{OWA}}h_{\text{MFPC}}(\mathbf{w},\mathbf{m},\mathbf{p}) = \sum_{i=1}^{n}\left(w_i \cdot \mu_{\text{MFPC},(i)}(m_i,\mathbf{p}_i)\right), \quad (10)$$

$$_{\text{AIWA}}h_{\text{MFPC}}(\mathbf{v},\mathbf{m},\mathbf{p}) = 1 - \left(\frac{\sum_{i=1}^{n}(1-\mu_{\text{MFPC},i}(m_i,\mathbf{p}_i))^{\frac{1}{\gamma}}}{n}\right)^{\gamma}, \text{ for } \rho\in\left[\tfrac{1}{2},1\right). \quad (11)$$

At dimension $n = 17$, thus aggregating 17 features, the geometric mean—and equally the MFPC—aggregation operator has a global andness $\rho_g^{h_{\text{GM}}}(17) = 0.637$. The OWA and AIWA operators are parameterized to have the geometric mean's andness and additionally some other andnesses to reveal their effects on the respective classification performance. Since these andnesses are from $\left[\frac{1}{2},1\right)$ (cf. Table I), (11) is given just for this interval.

To assure that all classifiers have the same fixed base of data to work with, all 17 used features are extracted only once prior to classification, stored and not changed afterwards. For each class, there exist 17 images which are used to train the classifiers in the learning phase. During this phase, all parameters needed to define the membership functions μ_{MFPC} are determined. For benchmarking, the classification rate $r_+ = \frac{n_+}{N}$ and—its dual—error rate $r_- = \frac{n_-}{N} = 1 - r_+$ of N objects to be classified are evaluated for each possible value of MFPC membership function's parameters p_{C_e} and D; n_+ and n_- are the numbers of correct and incorrect classifications, respectively. A classification is considered to be correct if the highest membership value μ belongs to the correct class and if additionally $\mu > 0.5$. Table I summarizes the classification rates of all classifier models. The table shows that specific parameter p_{C_e} of the MFPC membership function for which each classifier reaches its best classification rate r_+ per parameter D, grouped by the operators' andness degrees ρ_g. The best classification rate per group is printed bold. As reference, the MFPC classification performance is shaded gray.

Actually, the MFPC is performing worst amongst all evaluated classifier models, any other model leads to higher classification rates. The exception of this statement is $_{\text{AIWA}}h_{\text{MFPC}}$ at andness $\rho_g = 0.7$. This is caused due to many objects having membership values slightly below 0.5 and not being considered as classified correctly. A special case can be noticed at $\rho_g = 0.5$. In this case, both aggregation operators, OWA and AIWA, have completely identical properties and behave like the arithmetic mean. Compared to the best MFPC classification result, an improvement in classification rate of 10.72% can be achieved using the AIWA operator at an andness of $\rho_g = 0.56$ instead of using geometric mean aggregation for the described application. $\rho_g = 0.56$ is also the best in average, thus geometric mean's andness of $\rho_g^{h_{\text{GM}}}(n = 17) = 0.637$ tends to be too high.

V. CONCLUSION AND OUTLOOK

In this paper novel fuzzy pattern classifier models are derived based on the well established and studied Modified Fuzzy Pattern Classifier. It is shown that the MFPC incorporates a membership function and an aggregation operator, which is isolated and replaced. The replacement revealed substantial classification improvements. Other aggregation operators than those applied here as well as importance weighting may lead to additional improvements.

The investigantions presented shall be considered initial. Hence, no general statements can be derived from these and further experiments based on the initial findings are to be conducted. Nevertheless, the experiments presented in here lead the direction for further classification improvements, e.g. incorporating importance weights for aggregation.

Worthwhile investigating in the future is the threshold of 0.5 for the aggregated value μ, at which a classification is considered to be correct or not (cf. Sect. IV). Since operators with a higher andness will have a lower aggregated value of the same arguments, this should be taken into account by, e.g., setting the threshold to the operator's orness ω_g. Currently, the authors are working on substitutes of MFPC's membership function to investigate classification rate improvements by replacing this part of the classifier.

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